1 A general overview over the content of the School

Probably this text will be updated several times, suggestions are welcome

Arithmetic groups are groups of the form $\Gamma = \mathrm{Sl}_n(\mathbb{Z}), \mathrm{Sp}_g(\mathbb{Z})$... or more generally subgroups of finite index of those. They are per definitionem discrete subgroups of real Lie groups $G(\mathbb{R})$, for instance $\mathrm{Sl}_n(\mathbb{Z}) \subset \mathrm{Sl}_n(\mathbb{R})$. They act on the symmetric space $X = G(\mathbb{R})/K_\infty$, here K_∞ is a maximal compact subgroup, for example $K_\infty = \mathrm{SO}(n) \subset \mathrm{Sl}_n(\mathbb{R})$. The quotient spaces $\Gamma\backslash X$ are very interesting Riemannian manifolds (with possibly some "singularities").

We introduce sheaves $\tilde{\mathcal{M}}$ with values in finitely generated abelian groups, which are obtained from finitely generated Γ-modules $\mathcal{M}$. The case that

$$\Gamma = \mathrm{Sl}_2(\mathbb{Z}), X = \mathrm{Sl}_2(\mathbb{R})/\mathrm{SO}(2) = \text{ the upper half plane,}$$

and the Γ module

$$\mathcal{M}_n = \{\sum_{\nu=0}^{\nu=n} a_\nu X^\nu Y^{n-\nu}, a_\nu \in \mathbb{Z}\}$$

is an avatar of such an object.

The objects of interest are the sheaf-cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. By a general theorem of Raghunathan these cohomology groups are finitely generated abelian groups, but they have some extra structure:

a) The cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$ have a filtration which is induced by the non-compactness of $\Gamma\backslash X$.

b) These cohomology groups are modules for the so called Hecke algebra $\mathcal{H}$. This Hecke algebra contains a central subalgebra (the unramified Hecke algebra $\mathcal{H}_{\mathrm{un}}$) which is generated by Hecke operators $T_p^{(\chi)}$. (Here p runs over all "unramified" primes (depending on the choice of Γ there is a finite set of "ramified" primes) and χ runs over a finite set of cocharacters.)

bb) (Interlude) The structure of the Hecke algebra at an unramified prime is described by the Satake isomorphism. It says that the homomorphisms to a field F, i.e. $\pi_p : \mathcal{H}_{\mathrm{un}} \to F$ are in one to one correspondence to conjugacy classes $\omega(\pi_p) \in G^\vee(F)$ where $G^\vee$ is the Langlands dual group. In the case that the underlying group is GL_n the "Satake parameter" attached to a $\pi_p : \mathcal{H}_{\mathrm{un}} \to F$ is a diagonal element

$$\omega(\pi_p) = \begin{pmatrix} \omega_{1,p} & 0 & 0\dots & 0 & \\ 0 & \omega_{2,p} & 0 & \dots & \\ & & & & \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & 0 & \omega_{n,p} \end{pmatrix} \in \mathrm{GL}_n(\bar{F})$$

whose conjugacy class is invariant under the Galois group $\mathrm{Gal}(\bar{F}/F)$.

c) For any commutative ring R with identity we can consider the sheaf $\tilde{\mathcal{M}}_R = \tilde{\mathcal{M}} \otimes R$ and we can define the cohomology groups

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_R).$$

They still have the above filtration and are modules for the Hecke algebra.

This allows us to consider "eigenspaces"

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_F)(\pi_f) = \{x \in H^q(\Gamma\backslash X, \tilde{\mathcal{M}}_F) | T_p^{(\chi)}(x) = \pi_f(T_p^{(\chi)})x\}$$

where $\pi_f : \mathcal{H}_{\mathrm{un}} \to \mathcal{O}_F$ is a homomorphism and $\mathcal{O}_F$ is the ring of algebraic integers of a number field F.

To these eigenspaces we can attach the so called cohomological L-functions

$$L^{\mathrm{coh}}(\pi_f, r, s) = \prod_{p:prime} L^{\mathrm{coh}}(\pi_p, r, s).$$

Up to here the discussion is on a purely combinatorial level, the cohomology groups and the action of the Hecke operators can be computed from the Czech complex of a suitable finite acyclic covering of $\Gamma\backslash X$ by open sets.
(First section of lectures in the School)

For a deeper understanding of these L-functions we need tools from analysis (the theory of automorphic forms, representation theory and other things).
Under certain conditions these tools allow us to prove that these functions $L^{\mathrm{coh}}(\pi_f, r, s)$ are meromorphic (or even holomorphic) functions in the variable s and they satisfy a functional equation. (Whittaker models and L-functions). The Riemann ζ-function shows up in this context.

We will discuss the influence of the analytic properties of these L- functions on the structure of the cohomology groups (Eisenstein cohomology). The cohomological interpretation of these L-functions provides some rationality results for these L-functions at special arguments. Then we will investigate the influence of these values at special arguments on the structure of the cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$. (Second and third section of lectures in the school)

Again under some assumptions we can attach representations of the Galois group $\rho(\pi_f) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(M(\pi_f))$ to such eigenspaces. Eventually we will see that values of $L(\pi_f, r, s_0)$ at certain specific arguments have influence on the structure of the Galois group. (Last section of lectures)

2 Some more detailed explanations

2.1 The L-functions

After tensoring by a suitable algebraic number field $F \subset \mathbb{C}$ we may write a filtration

$$H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \supset \mathcal{F}^1 H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \supset \dots \mathcal{F}^\nu H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F) \dots \supset \{0\}$$

by $\mathcal{H}$ invariant subspaces such that the successive quotients

$$F^\nu H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F)/F^{\nu+1} H^q(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes F)$$

are absolutely irreducible $\mathcal{H}$ modules $H^q(\pi_{\nu,f})$. These subquotients are restricted tensor products

$$H^q(\pi_f) = \bigotimes_{p:prime}' H(\pi_p)$$

where $H(\pi_p)$ are absolutely irreducible $\mathcal{H}_p$ modules. At the unramified places such absolutely irreducible $\mathcal{H}_p$ modules are one dimensional and π_p is simply a homomorphism $\pi_p : \mathcal{H}_p \to \mathcal{O}_F$, hence they are determined by their values $\pi_f(T_p^{(i)})$.

To such an irreducible $\mathcal{H}$ module $H(\pi)$ and a second parameter r (an irreducible representation of the Langlands dual group) we can attach a (cohomological) L-function

$$L^{\mathrm{coh}}(\pi_f, r, s) = \prod_{p:prime} L^{\mathrm{coh}}(\pi_p, r, s)$$

where the local factors are of the form

$$L^{\mathrm{coh}}(\pi_p, r, s) = (1 - A_1(\pi_p, r)p^{-s} + \ldots A_d(\pi_p, r)p^{-ds})^{-1}$$

and where the $A_i(\pi_p, r) \in \mathcal{O}_F$, for unramified $\mathcal{H}_p$ the $A_i(\pi_p, r)$ are certain expressions in the $\pi(T_p^{(i)})$.

The products are convergent for $\Re(s) >> 0$. The Riemann $\zeta-$function occurs in this family of L-functions.

2.2 Relation to automorphic forms

The cohomology groups $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{C})$(the $^\bullet$ means we look at all degrees simultaneously) are related - via the Eichler-Shimura isomorphism - to automorphic forms. In other words we can apply analytic tools to get insight into the structure of the cohomology. For instance in certain cases certain subspaces of $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{C})$ can be identified with spaces of holomorphic modular forms.

We can use the theory of representations of $G(\mathbb{R})$ and Hodge-theoretic type of argument to prove the semi-simplicity of the so called "inner cohomology" under the action of the Hecke algebra, in certain cases we get formulas for the multiplicities $m(\pi)$ (multiplicity one theorems).

Finally the analytic theory of automorphic forms provides instruments for an understanding of the analytic properties of the L-functions $L(\pi, r, s)$ as functions in the variable s. Under certain assumptions it is possible to show that $L(\pi, r, s)$ extends to a meromorphic (or even holomorphic) function in the entire s-plane and satisfies a functional equation.

2.3 Eisenstein cohomology and special values

The spaces $\Gamma\backslash X$ are not compact in general, reduction theory tells us that we can describe some neighborhood $\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X)$ of infinity such that its complement in $\Gamma\backslash X$ is compact. This neighborhood of infinity is a union of open subsets

$$\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X) = \bigcup_P \overset{\bullet}{\mathcal{N}}_P(\Gamma\backslash X)$$

where P runs over the Γ- conjugacy classes of proper parabolic subgroups. We describe these pieces $\overset{\bullet}{\mathcal{N}}_P(\Gamma\backslash X)$ and their intersections in terms of fiber bundles over locally symmetric domains $\Gamma_H\backslash X^H$ attached to smaller reductive groups. This allows to compute the cohomology $H^\bullet(\overset{\bullet}{\mathcal{N}}_P(\Gamma\backslash X))$ in terms of the cohomology groups

$$H^\bullet(\Gamma_H\backslash X^H, \tilde{\mathcal{M}}(w))$$

where $\mathcal{M}(w)$ is a collection of Γ_H modules labeled by certain elements w in the Weyl group.

The goal of Eisenstein cohomology is to understand the restriction map

$$\ker(res) = H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \hookrightarrow H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}}) \xrightarrow{res} H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}}). \tag{1}$$

The right hand side contains pieces $H(\sigma) \subset H^\bullet(\Gamma_H\backslash X^H, \tilde{\mathcal{M}}(w))(\sigma) \subset H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})$ where σ is now an absolutely irreducible module for the Hecke algebra $\mathcal{H}_H$.

We want to understand how the "piece" $H(\sigma)$ is related to the image of res, for instance is $H(\sigma) \subset \mathrm{Im}(res)$?

a) We will explain that in certain situation the answer depends on w and whether a certain monomial expression

$$\mathcal{L}(\sigma, s) = \prod_r \frac{L(\sigma, r, m_r s)}{L(\sigma, r, m_r s + 1)}$$

is holomorphic at $s = 0$ or it has a pole.

b) We encounter situations where $\mathcal{L}(\sigma, s)$ is holomorphic and the cohomological interpretation yields a rationality result for special values of L-functions

$$\frac{1}{\Omega(\sigma)}\mathcal{L}(\sigma, 0) \in F^\times. \tag{2}$$

Here $\Omega(\sigma) \in \mathbb{C}^\times$ is a "period" which is well defined up to a unit in $\mathcal{O}_F^\times$ and which is obtained from the comparison of two different descriptions of the cohomology.

c) Once we have such a rationality result we may ask: What do these numbers tell us? We formulate some conjectures which roughly say that these have "influence" on the structure of the restriction map (1) considered as map between $\mathcal{H}$-modules. A somewhat very optimistic statement would be: If some power $\mathfrak{p}^{n_\mathfrak{p}}$ occurs in the prime factorization of the denominator of $\frac{1}{\Omega(\sigma)}\mathcal{L}(\sigma, 0)$ then we should find elements of order $\mathfrak{p}^{n_\mathfrak{p}}$ in

$$\xi(\sigma) \in \mathrm{Ext}^1_{\mathcal{H}}(H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}})(\sigma) \otimes \mathcal{O}_F/\mathfrak{p}^{n_\mathfrak{p}}, H_!^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathcal{O}_F/\mathfrak{p}^{n_\mathfrak{p}}) \tag{3}$$

This assertion is stronger than the statement that the Eisenstein class $\mathrm{Eis}(\sigma)$ has denominator $\mathfrak{p}^{n_\mathfrak{p}}$.

2.4 Galois-modules

There is a general idea, which goes back to several people and summarized under the name "Langlands philosophy", that to such an isotypical piece $H(\pi_f)$ should correspond a collection of "motives" $M(\pi, r)$ such that we have an equality of L-functions

$$L^{\mathrm{coh}}(\pi_f, r, s) = L(M(\pi, r), s) \tag{4}$$

This then implies that we should be able to attach to π (and some standard choice $r = r_0$ for the representation of the dual group) a compatible system of Galois representations

$$\{\rho_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{d(\pi)}(\mathbb{Q}_\ell)\} \tag{5}$$

such that we get an equality of L-functions. The existence of such a compatible system of Galois-modules has been proved in many cases.

The following case will be discussed in some detail. We assume that the quotient $\Gamma\backslash X$ is a Shimura variety, we even assume that $\Gamma\backslash X$ is in a certain natural way the set of complex valued points of a quasi projective variety $S/\mathrm{Spec}(\mathbb{Q})$. For any prime ℓ we can interpret $\mathcal{M} \otimes \mathbb{Z}_\ell$ as a sheaf for the etale topology on S and we can consider the etale cohomology groups

$$H^\bullet_{\text{ét}}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}}_\ell) = \lim_{\leftarrow} H^\bullet_{et}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m.\mathbb{Z}),$$

These cohomology groups are Galois-modules, and we have some control on ramification.

We may also read this as follows: We have the comparison isomorphism

$$H^\bullet_{et}(S \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{\sim} H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$$

and we use this to put the structure of a Galois module on $H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$, the Galois module structure commutes with the action of Hecke operators.

We can compactify $S \to S^\vee$ and this gives us a Galois-module structure on $H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z})$ as well and finally we get a $\mathcal{H} \times \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ invariant homomorphism

$$H^\bullet(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}) \xrightarrow{res} H^\bullet(\overset{\bullet}{\mathcal{N}}(\Gamma\backslash X), \tilde{\mathcal{M}} \otimes \mathbb{Z}/\ell^m\mathbb{Z}). \tag{6}$$

Hence we get from the extension classes in $\xi(\sigma)$ in (3) interesting extension of Galois-modules. (The prime ideal $\mathfrak{p}$ is now ℓ)